Chaotic Behavior in Stellar Dynamos

N. O. Weiss[1]

Slowly rotating main-sequence stars with deep convective zones have activity cycles like the sun's. The solar cycle is aperiodic and modulated to give intervals of reduced activity. A simple sixth-order system, obtained by truncating the dynamo equations, has solutions that mimic this behavior. The transition to chaos is analyzed and the astrophysical significance of these results is discussed.

KEY WORDS: Solar activity; magnetic cycles; dynamos; chaos.

1. INTRODUCTION

Sunspots have been studied since the time of Galileo and solar activity is known to vary cyclically with an average period of about 11 years. In the last decade, magnetic cycles have also been detected in a number of main-sequence stars. This solar–stellar connection allows us to use knowledge of the sun to explain stellar activity, and observations of stars to improve our understanding of the solar cycle.[1] There is a wealth of detailed information on the distribution of magnetic features over the surface of the sun. Such features cannot be resolved on stars but stellar observations do make it possible to explore the effect on magnetic activity of varying the rotation rate or the depth of the convective zone.

It is generally accepted that the sun's magnetic cycle is generated by a hydromagnetic dynamo located in, or at the base of, the convective zone.[2–4] Model equations, with periodic solutions that reproduce essential features of the solar cycle, can readily be constructed. The sunspot cycle is, however, aperiodic and there are intervals (such as the Maunder minimum in the late 17th century) when scarcely any spots appear. This pattern suggests that the solar dynamo behaves like a nonlinear oscillator in a

[1] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England, CB3 9EW.

regime where chaos has developed. Thus there is a need to relate dynamo theory to recent advances in the theory of nonlinear dynamical systems.

In what follows, I shall first review observations of solar and stellar activity; then I shall outline those aspects of dynamo theory that are relevant. In Section 4 I describe a simple nonlinear dynamo model that exhibits both periodic and chaotic oscillations.[5] The transition to chaos can be investigated by studying a reduced fifth-order system,[6] which is discussed in Section 5. Finally, the astrophysical significance of these results is summarized in the conclusion.

2. SOLAR AND STELLAR MAGNETIC ACTIVITY

The main features of the solar cycle have frequently been described[7] and they need only be summarized here. Sunspots appear at the beginning of a new cycle, at latitudes of $\pm 30°$; as the cycle progresses more sunspots occur at lower latitudes; then, toward the end of the cycle, the number of spots diminishes and the last spots appear near the equator, as the next cycle begins at higher latitudes. Thus activity appears as waves which migrate toward the equator with a period of about 11 years. Spots typically occur in pairs with opposite magnetic polarity, oriented parallel to the equator, and the sense of polarity is different in the northern and southern hemispheres. Moreover, the fields reverse after 11 years, so the magnetic cycle has a period of about 22 years.

Magnetic activity can be measured by computing the total area covered by sunspots or (more arbitrarily) by the sunspot number. The mean sunspot number (averaged over several rotations) varies aperiodically with time, with an average period that is well defined. On a longer time scale, there are episodes (such as the Spörer and Maunder minima in the 16th and 17th centuries) when sunspots almost completely disappear.[8] These grand minima correspond to periods of anomalous production of ^{14}C by cosmic rays and the incidence of grand minima over the past 5000 years can be derived from anomalies in ^{14}C dating.[9] It seems natural to explain this behavior as an example of chaos in a nonlinear dynamical system, with aperiodic oscillations modulated irregularly on a longer time scale.

Magnetic fields have been measured directly, using the Zeeman effect, in about 20 main sequence stars.[10] This group of stars is much more active than the sun, with fields comparable to that in a sunspot covering up to half their surfaces. These starspots are associated with flares and variations in luminosity. Thermal X-ray emission from stellar coronae provides indirect evidence of magnetic activity and the observations

obtained with the Einstein satellite show that there is more X-ray emission as the rotation rate is increased.[11]

The most systematic observations of stellar activity have been obtained at Mt. Wilson, using Ca^+ H and K emission, which is known to be correlated with magnetic fields on the sun. For stars of fixed spectral type, the Ca^+ emission increases monotonically with increasing rotation rate.[12] As a star evolves it loses angular momentum owing to magnetic braking: the magnetic field heats the corona and produces a stellar wind, which exerts a magnetic couple on the star. Thus the rotation period of a G star rises from about three days, shortly after its arrival on the main sequence, to 25 days at the age $(4.6 \times 10^9 \text{ yr})$ of the sun. In the young and rapidly rotating stars, magnetic activity is irregular and apparently acyclic (though any cycle might be masked by short-term fluctuations). Magnetic cycles resembling that in the sun have been identified in 12 older, more slowly rotating stars and, for a star of fixed mass, the cycle period increases approximately linearly with the rotation period.[13]

Linear (kinematic) dynamo models predict cyclic behavior but these observations raise several important questions for the theoretician. What mechanism limits dynamo action in a nonlinear regime? How does aperiodicity develop in such a system? Can the grand minima be explained? And why is there no clear evidence for cycles in rapidly rotating stars? To answer these questions we must consider dynamo theory in more detail.

3. DYNAMO MODELS

The basic principles of dynamo action are the production of toroidal flux from a poloidal field through differential rotation and regeneration of a reversed poloidal field owing to helicity.[2,3] To describe this process we must solve the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \text{curl}(\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{1}$$

and the equation of motion

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = \rho \mathbf{g} - \nabla p + \mathbf{j} \times \mathbf{B} + \rho \nu \nabla^2 \mathbf{u} \tag{2}$$

together with an energy equation, for the velocity $\mathbf{u}$ and the magnetic field $\mathbf{B}$ (where other symbols are defined as usual). Large-scale numerical simulations by Gilman,[14] for a Boussinesq fluid, and Glatzmaier,[15] in the

anelastic approximation, have produced a convincing description of the angular momentum distribution in the convective zone. They find that the angular velocity, Ω, decreases inward, in agreement with values derived from the splitting of frequencies of solar oscillations.[16] Magnetic cycles can indeed be generated but the dynamo waves travel poleward, in the wrong direction, as Glatzmaier[15] will explain.

An easier approach is to obtain mean field equations by averaging azimuthally and parametrizing the difficult nonlinear effects. Then we may separate the (axisymmetric) poloidal and toroidal mean fields by writing

$$\mathbf{B} = \mathbf{B}_P + B\hat{\boldsymbol{\phi}}, \qquad \mathbf{B}_P = \mathrm{curl}(A\hat{\boldsymbol{\phi}}) \tag{3}$$

where $\hat{\boldsymbol{\phi}}$ is a unit vector in the azimuthal direction. From (1)

$$\frac{\partial A}{\partial t} = \alpha B + \eta \mathscr{D}^2 A \tag{4}$$

$$\frac{\partial B}{\partial t} = r \sin \theta \mathbf{B}_P \cdot \nabla\Omega + \eta \mathscr{D}^2 B \tag{5}$$

where the operator $\mathscr{D}^2 = \nabla^2 - 1/r^2 \sin^2 \theta$. Equations (4) and (5) describe an $\alpha\Omega$-dynamo, with the effects of helicity represented by α, and an enhanced turbulent diffusivity η. For a given geometry there is a single stability parameter, the dynamo number

$$D = \alpha\Omega' L^4/\eta^2 \tag{6}$$

where Ω' measures the angular velocity gradient and L is a typical length. As D is increased, the trivial solution $\mathbf{B} = 0$ loses stability at a Hopf bifurcation. Solutions to this linear problem give dynamo waves that propagate toward the equator—provided that Ω *increases* inward. Since Ω actually seems to *decrease* inward through most of the convective zone, it is tempting to suppose that the dynamo is located in a thin shell at the interface between the radiative and convective zones, an assumption that has several advantages.[4]

In a nonlinear dynamo growth of the magnetic field is limited by the effect of the Lorentz force on the motion. This may lead to quenching of the α effect or to changes in the azimuthal velocity v. Then we may write the averaged azimuthal component of (2) as

$$\frac{\partial v}{\partial t} = F(\mathbf{r}) + \rho^{-1}\mathbf{j} \times \mathbf{B} \cdot \hat{\boldsymbol{\phi}} + v\mathscr{D}^2 v \tag{7}$$

where turbulent transport of angular momentum is parametrized in terms of a volume force ρF and a diffusivity v. If we write $v = \mathbf{V}(r) + w(\mathbf{r}, t)$, where the mean flow V driven by convection satisfies

$$F + v\mathscr{D}^2 V = 0 \tag{8}$$

and the Lorentz force drives a fluctuating velocity w such that

$$\frac{\partial w}{\partial t} = \frac{1}{\mu_0 \rho} \left[\nabla \times (B\hat{\boldsymbol{\phi}}) \right] \times \mathbf{B}_P \cdot \hat{\boldsymbol{\phi}} + v\mathscr{D}^2 w \tag{9}$$

then we expect w to vary with twice the frequency of the magnetic cycle. Such torsional waves have indeed been observed in the sun.[17]

A more drastic approach is to replace the mean field equations (4), (5), and (9) by some truncated system of coupled nonlinear differential equations whose properties can be investigated in some detail. This procedure is somewhat dubious but has obvious advantages. Indeed, chaotic behavior was found as long ago as 1962 for the third-order system governing a pair of coupled disk dynamos[18,19] introduced to model reversals of the earth's magnetic field.[20] More recently, hyperchaos has been found for three disk dynamos, governed by a sixth-order system.[21] The geodynamo maintains a more or less steady field for intervals much longer than the ohmic decay time for the earth's core, whereas the period of a stellar magnetic cycle is far smaller than the ohmic decay time. In the next section I shall describe a simple system that models the transition to chaos for an oscillatory dynamo.

4. NONLINEAR DYNAMO WAVES

Following Parker,[3] we consider dynamo waves in a thin spherical shell and adopt local cartesian coordinates with the z axis pointing radially outward and the y axis pointing westward. Then we consider plane waves propagating in the x direction, such that

$$\mathbf{B} = (0, B(t)\, e^{ikx}, ikA(t)\, e^{ikx}) \tag{10}$$

in the presence of an azimuthal velocity $\mathbf{v} = (V(z) + w(x, z, t))\hat{\mathbf{y}}$. Now Eqs. (4) and (5) can be rewritten in dimensionless form as

$$\dot{A} = 2DB - A, \qquad \dot{B} = iA - B \tag{11}$$

where the dynamo number $D = \alpha V'/2\eta^2 k^3$, and the trivial solution $A = B = 0$ undergoes a Hopf bifurcation at $D = 1$. For $D > 1$, Eq. (11) has

unstable oscillatory solutions, corresponding to one-dimensional dynamo waves.[3]

Growth of these solutions is limited by nonlinear effects. For instance, we might suppose that the α effect is reduced as the field increases, so that

$$\dot{A} = 2D(1 + \kappa |B|^2)^{-1} - A \tag{12}$$

or that magnetic buoyancy leads to an enhanced dissipation of toroidal flux, so that

$$\dot{B} = iA - (1 + \lambda |B|^2)B \tag{13}$$

where κ, λ are positive constants. Equations (12) and (13) possess stable, periodic nonlinear solutions.[5] Alternatively, we may consider the effect of the Lorentz force on the sheared velocity, setting

$$\frac{\partial w}{\partial z} = \omega_0(t) + \omega(t) \exp 2ikx \tag{14}$$

so that (4), (5), and (9) can be written in the form

$$\dot{A} = 2DB - A, \qquad \dot{B} = i(1 + \omega_0)A - \tfrac{1}{2}iA^*\omega - B \tag{15}$$

$$\dot{\omega}_0 = \tfrac{1}{2}i(A^*B - AB^*) - v_0\omega_0, \qquad \dot{\omega} = -iAB - v\omega \tag{16}$$

where ω_0 is real but ω, like A and B, is complex, and v_0, v are constants. The seventh-order system (15)–(16) again possesses an exact periodic solution. It is convenient to consider two limiting cases. In the first, we let $v \to \infty$, so that $\omega \to 0$ and we obtain a *fifth*-order system. The Lorentz force slows down differential rotation, reducing the effective value of the dynamo number, until an equilibrium is reached. This process models behavior found in Gilman's computations.[14] The second limit is obtained by letting $v_0 \to \infty$, $\omega_0 \to 0$, so that (15)–(16) reduce to a *sixth*-order system, and dynamo action is limited by fluctuations in differential rotation.

Observations of stellar magnetic cycles can be used to eliminate some of these models.[13] We expect that $D \propto \Omega^2$, since the α effect depends linearly on the angular velocity through the Coriolis force and differential rotation should also vary linearly with Ω in a slowly rotating star. We therefore require that the toroidal field strength, $|B|$, should increase and that the cycle period should decrease with increasing D, in order to match the observations. For the fifth-order system, however, $|B|$ decreases monotonically for $D > 2$ and the cycle period is constant; quenching of the α effect, as in (12), does give an increasing magnetic field but the period is still constant. Thus both these processes (which correspond to reducing the

effective value of D) can be ruled out. On the other hand, limitation either through enhanced losses, as in (13), or owing to fluctuations in differential rotation, is compatible with observations. The latter process also leads to chaotic behavior resembling that found in the sun and solutions of the sixth-order system have therefore been studied in some detail.[5]

This nonlinear system possesses an exact periodic solution for $D > 1$, with a frequency $p(v, D)$ such that A and B vary as exp ipt, while ω varies as exp $2ipt$, as in the torsional waves observed on the sun.[17] As D is increased, the periodic solution becomes unstable if $v < 1$. When $v = 0.5$, there is a Hopf bifurcation at $D \approx 2.07$, leading to doubly periodic behavior. Figure 1a shows a trajectory for $D = 2.6$, projected onto the $B_1 \omega_1$ plane, where $B_1 = \mathbb{R}eB$, etc., and B_1 is shown as a function of time in Fig. 1b. Evidently the trajectory lies on a two-torus in phase space. At $D \approx 3.47$ there is a further Hopf bifurcation, leading to triply periodic motion, followed by a transition to chaos around $D = 3.84$. Figure 2 shows irregular behavior found for $D = 8.0$. The trajectory wanders chaotically, spending intervals in the neighborhood of the origin in phase space. The magnetic field oscillates aperiodically and is modulated to give episodes of reduced activity. This plot bears a strong qualitative resemblance to the record of solar activity, with aperiodic magnetic cycles modulated to give grand minima. So the most obvious nonlinear features of the solar cycle can be reproduced by a simple sixth-order system of equations. In order to assess the significance of these results we must analyze the bifurcation sequence in more detail.

5. QUASIPERIODICITY AND CHAOS

The system (15)–(16) possesses a symmetry under the transformation $(A, B, \omega) \rightarrow (Ae^{i\psi}, Be^{i\psi}, \omega e^{2i\psi})$, corresponding to symmetry with respect to translation in the x direction. It is therefore possible to reduce its order by making the transformation

$$A = 2\rho^{1/2}e^{i\theta}, \qquad B = D^{-1}\rho^{1/2}xe^{i\theta}, \qquad \omega = D^{-1}ye^{2i\theta} \qquad (17)$$

Then θ is ignorable and the sixth-order system [obtained by setting $v_0 = \infty$, $\omega_0 = 0$ in (16)] reduces to the fifth-order system[6]

$$\dot{\rho} = \rho(x + x^*) - 2\rho$$
$$\dot{x} = 2iD - iy - x^2, \qquad \dot{y} = -2i\rho x - y(x - x^*) - vy \qquad (18)$$

This system has two "trivial" fixed points, at $\rho = y = 0$, $x = \pm(1 + i) D^{1/2}$, which we shall call O_1, O_2, respectively. O_2 is unstable for all positive

(a)
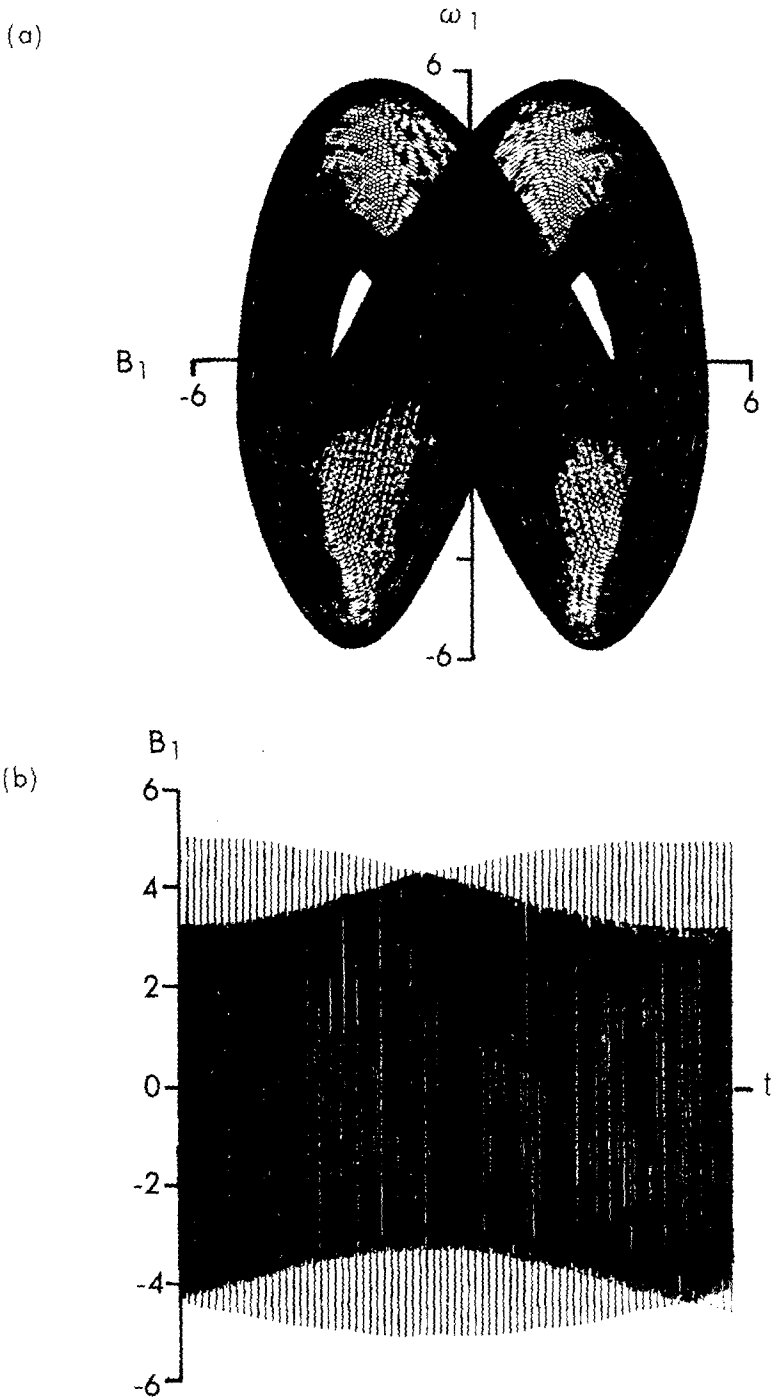
ω_1

B_1

(b)

B_1

Fig. 1. Solutions of the sixth-order system for $D = 2.6$. (a) Trajectory projected onto the $B_1\omega_1$ plane. (b) B_1 as a function of time. The trajectory lies on a torus and B_1 is doubly periodic.
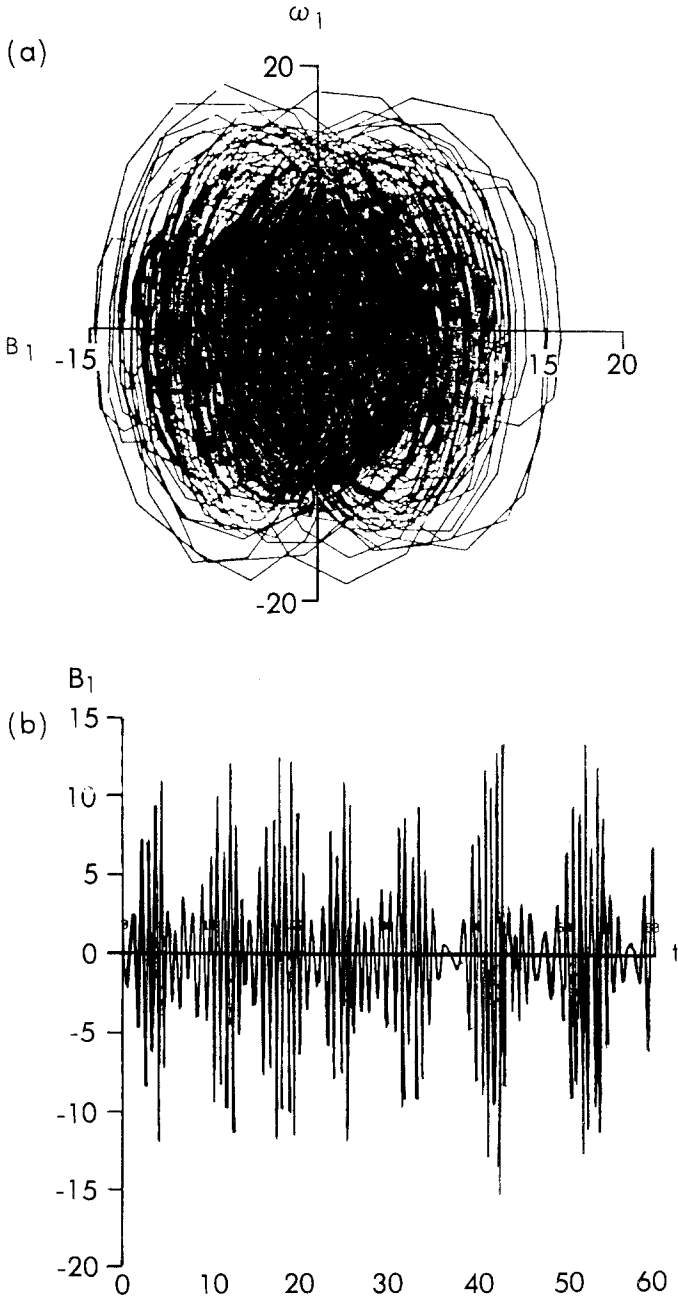
Fig. 2.   As Fig. 1 but for $D = 8.0$. The solution is chaotic and shows intervals of reduced activity when the trajectory hovers near the origin.

values of D but its unstable manifold (the plane $\rho = y = 0$) is contained in the stable manifold of O_1. As D is increased through unity, a real eigenvalue at O_1 passes through zero. Thus the Hopf bifurcation in (11) becomes a simple bifurcation for (18) and the limit cycle is replaced by a nontrivial fixed point.

Subsequent bifurcations have been followed in detail for the case $v = 0.5$.[6] At $D \approx 2.07$ there is a Hopf bifurcation, followed by the appearance of a limit cycle (corresponding to a two-torus in the original sixth-order system) which sheds a two-torus (corresponding to the triply periodic motion mentioned in the previous section) when $D \approx 3.47$. Figure 3 shows a trajectory for $D = 3.5$, projected (for convenience of representation) onto the rz_1 plane, where $r = 2\rho^{1/2}$ and $z = rx$, together with a Poincaré section in the same plane for $y_1 = -2$. Apparently the trajectory lies on a torus. As D is increased, frequency locking occurs: Figure 4 shows a periodic solution for $D = 3.8$, which repeats exactly after winding 25 times round the torus. At $D \approx 3.806$ there is a period-doubling bifurcation, followed by periodic trajectories which repeat after 50 cycles, like that in Figure 5. This is followed by a Feigenbaum cascade of bifurcations, accumulating at $D \approx 3.84$. Thereafter, solutions seem to be chaotic, except for narrow windows where periodic solutions can be identified.

Inspection of Figs. 3–5 shows that trajectories approach closer to the origin as D is increased. It has therefore been conjectured[6] that the unstable periodic solution with winding number 1/25 continues to exist for all $D > 3.806$ and that, in the limit $D \to \infty$, there is an unstable heteroclinic orbit connecting the two singular points O_1 and O_2 of the system (18). At O_1 the eigenvalues are $-2(1 + i) D^{1/2}$, $-(v + 2iD^{1/2})$, and $(D^{1/2} - 1)$, the first is complex and strongly contracting, the second is complex and stable, and the third is real and strongly unstable. The last two satisfy Shilnikov's criterion and local analysis suggests that chaotic behavior should be found for D sufficiently large.[22] At O_2, on the other hand, the eigenvalues are $-(D^{1/2} + 1)$, $-v + 2iD^{1/2}$ and $2(1 + i) D^{1/2}$: the first is real and strongly contracting, while the other two are complex but have identical imaginary parts. It is not obvious whether such eigenvalues lead to chaos. We do, however, expect to find chaos associated with the elaborate heteroclinic trajectory connecting O_1 and O_2 with $\rho \neq 0$ and then returning from O_2 to O_1 in the plane $\rho = y = 0$.

6. CONCLUSION

The model system (15)–(16) can be derived as a truncation of the mean field dynamo equations. There is, however, no rigorous justification for adopting the latter equations in a star. Still less can we be certain that
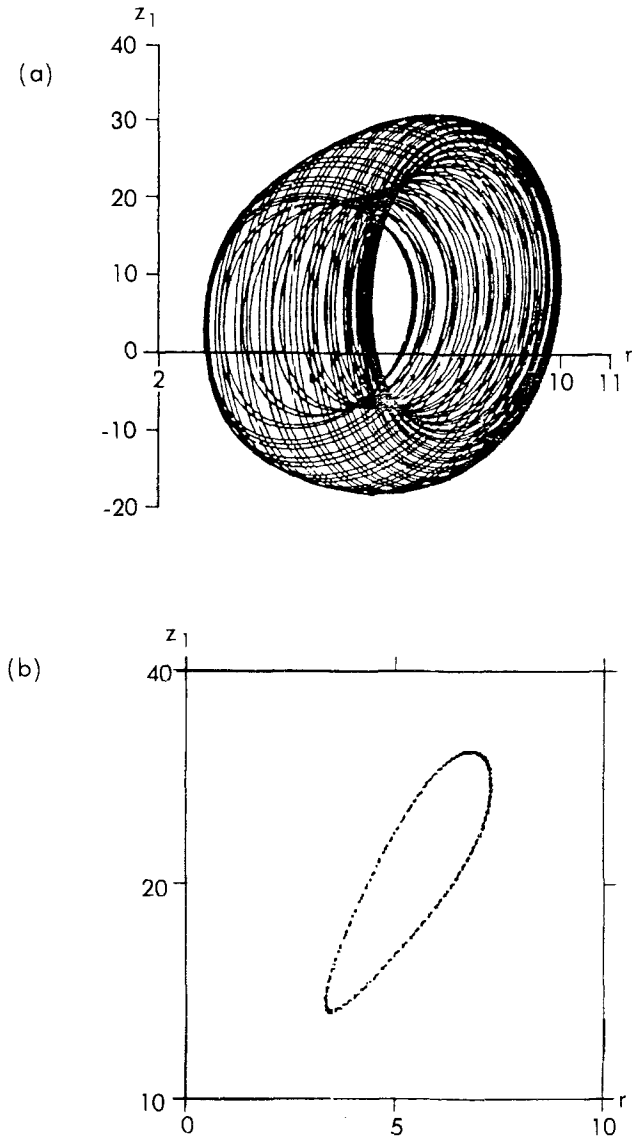
Fig. 3. Solutions of the reduced fifth order system (18) for $D = 3.5$. (a) Trajectory projected onto the rz_1 plane. (b) Poincaré section in the rz_1 plane for $y_1 = -2.0$. The trajectory lies on a torus, whose cross section is a closed curve.
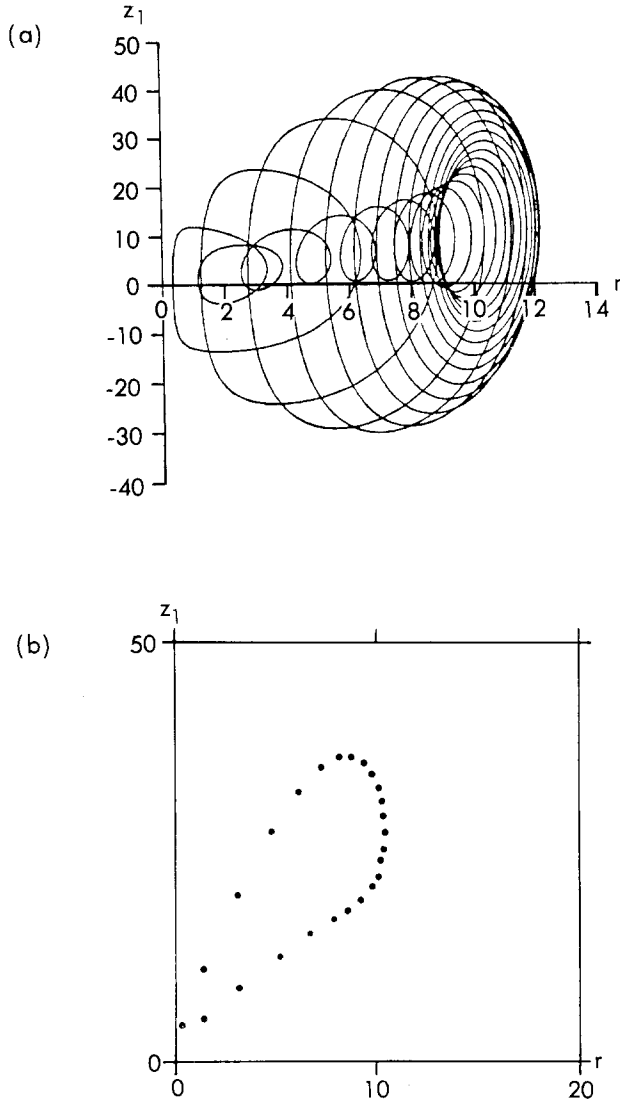
Fig. 4.   As Fig. 3 but for $D = 3.8$. The solution is periodic and repeats exactly after 25 cycles. The return map shows 25 discrete points.
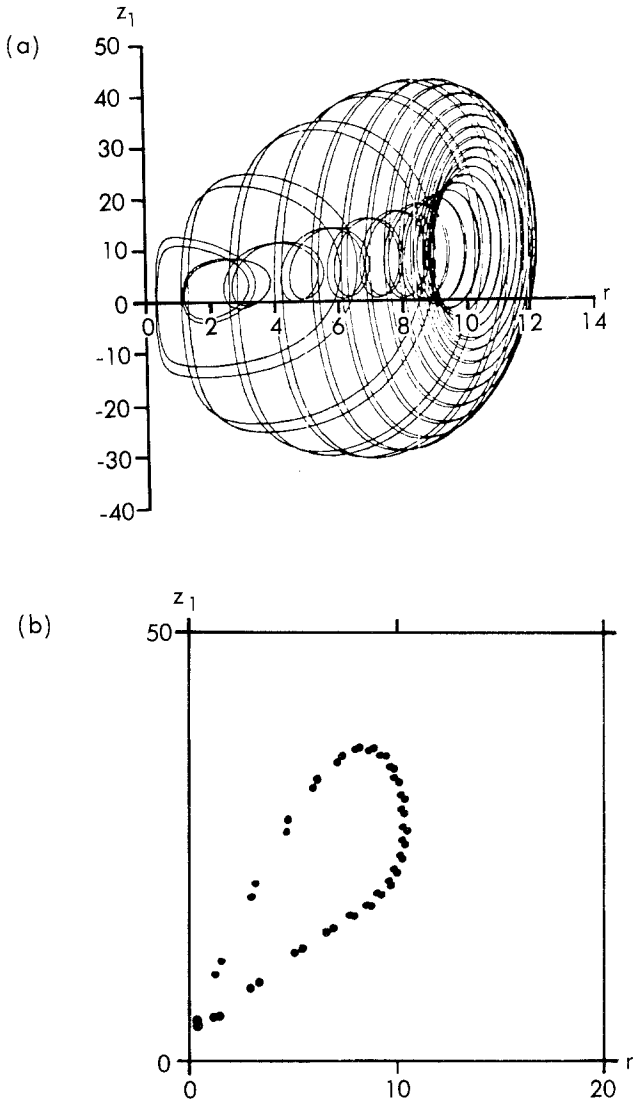
Fig. 5. As Fig. 3 but for $D = 3.81$. After the first period doubling bifurcation the solution repeats after winding twice round the torus.

the bifurcation structure found for a low-order, arbitrarily truncated system applies also to the partial differential equations from which it was derived. Nevertheless, the model system does include the relevant physics and yields solutions that mimic the observed properties of magnetic cycles in the sun and stars. With all its limitations, it provides some clues to the behavior of nonlinear stellar dynamos.

First of all, we have established that a simple nonlinear model with dynamical coupling can produce both aperiodic cycles and episodes of reduced activity. The behavior of the solar cycle can be explained as a consequence of deterministic chaos without invoking stochastic disturbances. Furthermore, the grand minima are apparently associated with the persistence of a "ghost" attractor with a small winding number in the chaotic regime. Indeed, the envelope of activity (as determined from ^{14}C anomalies) shows similar structure around different grand minima.[9] Thus the record of solar activity suggests a transition through quasiperiodicity to chaos as the dynamo number (or the angular velocity of a star) is increased.

Such a bifurcation sequence is not peculiar to the system that we have considered and it is not difficult to construct third-order systems that exhibit similar behavior.[22,23] So it is likely that other physical mechanisms (such as instabilities associated with magnetic buoyancy[24]) could produce a similar pattern of activity. We have already seen how simple models can be used to isolate the most significant nonlinear processes in a stellar dynamo. When used in conjunction with self-consistent simulations,[25] such models make it possible to recognize the bifurcation structure in quite complicated problems.

ACKNOWLEDGMENTS

REFERENCES

1. J. O. Stenflo, ed., *Solar and Stellar Magnetic Fields* (Reidel, Dordrecht, 1983).
2. H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, 1978).
3. E. N. Parker, *Cosmical Magnetic Fields* (Clarendon Press, Oxford, 1979).
4. N. O. Weiss, in *Stellar and Planetary Magnetism*, A. M. Soward, ed. (Gordon & Breach, London, 1983), p. 115.
5. N. O. Weiss, F. Cattaneo, and C. A. Jones, *Geophys. Astrophys. Fluid Dyn.* **30**:305 (1984).

6. C. A. Jones, N. O. Weiss, and F. Cattaneo, *Physica* **14D**:161 (1985).
7. R. W. Noyes, *The Sun, our Star* (Harvard University Press, Cambridge, 1982).
8. J. A. Eddy, *Science* **286**:1189 (1976).
9. M. Stuiver, *Nature (London)* **286**:868 (1980); M. Stuiver and P. D. Quay, *Science* **207**:11 (1980).
10. G. D. Marcy, *Astrophys. J.* **276**:286 (1984).
11. G. S. Vaiana *et al.*, *Astrophys. J.* **245**:163 (1981); R. Pallavicini, L. Golub, R. Rosner, G. Vaiana, T. Ayres, and J. Linsky, *Astrophys. J.* **248**:279 (1981).
12. R. W. Noyes, L. W. Hartmann, S. L. Baliunas, D. K. Duncan, and A. H. Vaughan, *Astrophys. J.* **279**:763 (1984).
13. R. W. Noyes, N. O. Weiss, and A. H. Vaughan, *Astrophys. J.* **287**:769 (1984).
14. P. A. Gilman, *Astrophys. J. Suppl. Ser.* **53**:243 (1983).
15. G. A. Glatzmaier, this issue (1985).
16. T. L. Duvall and J. W. Harvey, *Nature (London)* **310**:19 (1984); T. L. Duvall, W. A. Dziembowski, P. R. Goode, D. O. Gough, J. W. Harvey, and J. W. Leibacher, *Nature (London)* **310**:22 (1984).
17. R. F. Howard and B. J. LaBonte, *Astrophys. J.* **239**:L33 (1980); B. J. LaBonte and R. F. Howard, *Solar Phys.* **75**:161 (1982).
18. D. W. Allan, *Proc. Cambridge Philos. Soc.* **58**:671 (1962).
19. K. Ito, *Earth Planet. Sci. Lett.* **51**:451 (1980).
20. E. C. Bullard, in *Topics in Nonlinear Dynamics*, S. Jorna, ed. (American Institute of Physics, New York, 1978), p. 373.
21. T. Miura and T. Kai, *Phys. Lett.* **101A**:450 (1984).
22. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 1983).
23. W. F. Langford, in *Nonlinear Dynamics and Turbulence*, G. I. Barenblatt, G. Iooss, and D. D. Joseph, eds. (Pitman, London, 1983), p. 215.
24. S. Childress and E. A. Spiegel, in *Variations of the Solar Constant*, S. Sofia, ed. (NASA, Washington, D.C., 1981), p. 273.
25. D. R. Moore, J. Toomre, E. Knobloch, and N. O. Weiss, *Nature (London)* **303**:663 (1983).